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AUTHOR(S):

Kamimura, Shoji; Khan, Safeer Hussain; Takahashi,
Wataru

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ITERATIVE SCHEMES FOR APPROXIMATING SOLUTIONS OF RELATIONS INVOLVING ACCRETIVE OPERATORS IN BANACH SPACES

SHOJI KAMIMURA, SAFEER HUSSAIN KHAN AND WATARU TAKAHASHI

ABSTRACT. In this paper, we introduce two iterative schemes for approximating solutions of the relation $0 \in Av$, where A is an accretive operator satisfying the range condition.

1. INTRODUCTION

Let E be a real Banach space, let $A \subset E \times E$ be an m -accretive operator and let $J_r = (I + rA)^{-1}$ be the resolvent of A for $r > 0$. In this paper, we shall study iterative schemes for solving the relation $0 \in Av$. A well-known method is the following: $x_0 = x \in E$,

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where $\{r_n\}$ is a sequence of positive real numbers. The convergence of (1.1) has been studied by Rockafellar [15], Brézis and Lions [1], Lions [7], Pazy [11], Bruck and Reich [4], Reich [12, 13], Nevanlinna and Reich [9], Bruck and Passty [3], Jung and Takahashi [6] etc. On the other hand, Halpern [5] and Mann [8] introduced the following iterative schemes for approximating fixed points of nonexpansive mappings T of E into itself:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \dots \quad (1.2)$$

and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 0, 1, 2, \dots, \quad (1.3)$$

respectively, where $x_0 = x \in E$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. The iterative schemes (1.2) and (1.3) have been studied extensively. See, for example, Takahashi [18, 19] and the references therein.

In this paper, motivated by (1.1), (1.2) and (1.3), we study two iterative schemes to solve the relation $0 \in Av$, where A is an accretive operator satisfying the range condition, that is, $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Then correspondence to (1.2) is

$$x_{n+1} = P(\alpha_n x + (1 - \alpha_n)J_{r_n} x_n + f_n), \quad n = 0, 1, 2, \dots$$

and that to (1.3) is

$$x_{n+1} = P(\alpha_n x_n + (1 - \alpha_n)J_{r_n} x_n + f_n), \quad n = 0, 1, 2, \dots,$$

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where P is a nonexpansive retraction of E onto C and f_n is the term showing a computational error.

2. PRELIMINARIES

Throughout this paper, we denote the set of all nonnegative integers by \mathbb{N} . Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E . We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E , we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The modulus of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then δ satisfies that

$$\left\| \frac{x+y}{2} \right\| \leq r \left(1 - \delta \left(\frac{\epsilon}{r} \right) \right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \epsilon$. Let $U = \{x \in E : \|x\| = 1\}$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for every $x \in E$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \quad (2.1)$$

is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E . A Banach space E is said to satisfy Opial's condition [10] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n \rightarrow \infty} \|x_n - y\| < \liminf_{n \rightarrow \infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$.

Let C be a closed convex subset of E . A mapping $T: C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D . Let D be a subset of C . We denote the closure of the convex hull of D by $\overline{\text{co}}D$. A mapping P of D into itself is said to be a retraction if $P^2 = P$. A subset D of C is said to be a nonexpansive retract of C if there exists a nonexpansive retraction of C onto D .

Let I denote the identity operator on E . An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup \{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If A is accretive, then we have $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$ and $r > 0$. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued

mapping $J_r: R(I + rA) \rightarrow D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A . We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator A satisfying the range condition, $A^{-1}0 = F(J_r)$ for all $r > 0$. An accretive operator A is said to be m -accretive if $R(I + rA) = E$ for all $r > 0$.

In the sequel, unless stated otherwise, we assume that $A \subset E \times E$ is an accretive operator satisfying the range condition and that J_r is the resolvent of A for $r > 0$.

3. STRONG CONVERGENCE THEOREM

In this section, we study the strong convergence of Halpern's type iteration. We need the following result for the proof of our theorem.

Theorem 1 (Takahashi and Ueda [21]). *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. If $A^{-1}0 \neq \emptyset$, then the strong $\lim_{t \rightarrow \infty} J_t x$ exists and belongs to $A^{-1}0$ for all $x \in C$.*

See also Reich [14]. Using this result, we prove the following theorem. The proof is mainly due to Wittmann [22] and Shioji and Takahashi [16].

Theorem 2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex nonexpansive retract of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and let P be a nonexpansive retraction of E onto C . Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P(\alpha_n x + (1 - \alpha_n)J_{r_n} x_n + f_n), \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} r_n = \infty$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

Proof. Let $y_n = J_{r_n} x_n$, $v_n = \alpha_n x + (1 - \alpha_n)y_n + f_n$ and $u \in A^{-1}0$. Then we have

$$\begin{aligned} \|x_1 - u\| &= \|P(\alpha_0 x + (1 - \alpha_0)y_0 + f_0) - Pu\| \\ &\leq \|\alpha_0 x + (1 - \alpha_0)y_0 + f_0 - u\| \\ &\leq \alpha_0 \|x - u\| + (1 - \alpha_0)\|y_0 - u\| + \|f_0\| \\ &\leq \alpha_0 \|x - u\| + (1 - \alpha_0)\|x_0 - u\| + \|f_0\| \\ &= \|x - u\| + \|f_0\|. \end{aligned}$$

If $\|x_n - u\| \leq \|x - u\| + \sum_{i=0}^{n-1} \|f_i\|$ for some $n \in \mathbb{N} \setminus \{0\}$, then we can similarly show that $\|x_{n+1} - u\| \leq \|x - u\| + \sum_{i=0}^n \|f_i\|$. Therefore, by induction, we obtain $\|x_{n+1} - u\| \leq \|x - u\| + \sum_{i=0}^n \|f_i\|$ for all $n \in \mathbb{N}$ and hence $\{x_n\}$ is bounded because $\sum_{n=0}^{\infty} \|f_n\| < \infty$. Then $\{y_n\}$ and $\{v_n\}$ are also bounded. Next we shall show that

$$\limsup_{n \rightarrow \infty} \langle x - z, J(v_n - z) \rangle \leq 0. \quad (3.1)$$

Since $(x - J_t x)/t \in AJ_t x$, $A_{r_n} x_n \in Ay_n$ and A is accretive, we have

$$\left\langle A_{r_n} x_n - \frac{x - J_t x}{t}, J(y_n - J_t x) \right\rangle \geq 0$$

and hence

$$\langle x - J_t x, J(y_n - J_t x) \rangle \leq t \langle A_{r_n} x_n, J(y_n - J_t x) \rangle$$

for all $n \in \mathbb{N}$ and $t > 0$. Then, from $A_{r_n} x_n = (x_n - y_n)/r_n \rightarrow 0$ as $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} \langle x - J_t x, J(y_n - J_t x) \rangle \leq 0 \quad (3.2)$$

for all $t > 0$. It follows from Theorem 1 that $J_t x \rightarrow z \in A^{-1}0$ as $t \rightarrow \infty$. Then, since the norm of E is uniformly Gâteaux differentiable, for any $\epsilon > 0$, there exists $t_0 > 0$ such that

$$|\langle z - J_t x, J(y_n - J_t x) \rangle| \leq \frac{\epsilon}{2} \quad \text{and} \quad |\langle x - z, J(y_n - J_t x) - J(y_n - z) \rangle| \leq \frac{\epsilon}{2}$$

for all $t \geq t_0$ and $n \in \mathbb{N}$. Then it follows that

$$\begin{aligned} & |\langle x - J_t x, J(y_n - J_t x) \rangle - \langle x - z, J(y_n - z) \rangle| \\ & \leq |\langle x - J_t x, J(y_n - J_t x) \rangle - \langle x - z, J(y_n - J_t x) \rangle| \\ & \quad + |\langle x - z, J(y_n - J_t x) \rangle - \langle x - z, J(y_n - z) \rangle| \\ & = |\langle z - J_t x, J(y_n - J_t x) \rangle| + |\langle x - z, J(y_n - J_t x) - J(y_n - z) \rangle| \\ & \leq \epsilon \end{aligned} \quad (3.3)$$

for all $t \geq t_0$ and $n \in \mathbb{N}$. Therefore it follows from (3.2) and (3.3) that

$$\limsup_{n \rightarrow \infty} \langle x - z, J(y_n - z) \rangle \leq \limsup_{n \rightarrow \infty} \langle x - J_t x, J(y_n - J_t x) \rangle + \epsilon \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\limsup_{n \rightarrow \infty} \langle x - z, J(y_n - z) \rangle \leq 0. \quad (3.4)$$

On the other hand, since $v_n - y_n = \alpha_n(x - y_n) + f_n \rightarrow 0$ as $n \rightarrow \infty$ and the norm of E is uniformly Gâteaux differentiable, we have

$$\lim_{n \rightarrow \infty} |\langle x - z, J(v_n - z) \rangle - \langle x - z, J(y_n - z) \rangle| = 0. \quad (3.5)$$

Combining (3.4) and (3.5), we obtain (3.1).

From $(1 - \alpha_n)(y_n - z) = (v_n - z) - \alpha_n(x - z) - f_n$, we have

$$(1 - \alpha_n)^2 \|y_n - z\|^2 \geq \|v_n - z\|^2 - 2\langle \alpha_n(x - z) + f_n, J(v_n - z) \rangle$$

and hence

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|Pv_n - Pz\|^2 \leq \|v_n - z\|^2 \\ &\leq (1 - \alpha_n)^2 \|y_n - z\|^2 + 2\langle \alpha_n(x - z) + f_n, J(v_n - z) \rangle \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n \langle x - z, J(v_n - z) \rangle + M \|f_n\| \end{aligned}$$

for all $n \in \mathbb{N}$, where $M = 2 \sup_{n \in \mathbb{N}} \|v_n - z\|$. By (3.1) and $\sum_{n=0}^{\infty} \|f_n\| < \infty$, for any $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$M \sum_{i=m}^{\infty} \|f_i\| \leq \frac{\epsilon}{2} \quad \text{and} \quad \langle x - z, J(v_n - z) \rangle \leq \frac{\epsilon}{2}$$

for all $n \geq m$. Hence

$$\|x_{n+m+1} - z\|^2 \leq (1 - \alpha_{n+m}) \|x_{n+m} - z\|^2 + \alpha_{n+m} \frac{\epsilon}{2} + M \|f_{n+m}\|$$

for all $n \in \mathbb{N}$. Then, by induction, we obtain

$$\begin{aligned} \|x_{n+m+1} - z\|^2 &\leq \|x_m - z\|^2 \prod_{i=m}^{n+m} (1 - \alpha_i) + \left\{ 1 - \prod_{i=m}^{n+m} (1 - \alpha_i) \right\} \frac{\epsilon}{2} + M \sum_{i=m}^{n+m} \|f_i\| \\ &\leq \|x_m - z\|^2 \exp \left(- \sum_{i=m}^{n+m} \alpha_i \right) + \frac{\epsilon}{2} + M \sum_{i=m}^{n+m} \|f_i\| \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore it follows from $\sum_{n=0}^{\infty} \alpha_n = \infty$ that

$$\limsup_{n \rightarrow \infty} \|x_n - z\|^2 = \limsup_{n \rightarrow \infty} \|x_{n+m+1} - z\|^2 \leq \frac{\epsilon}{2} + M \sum_{i=m}^{\infty} \|f_i\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\{x_n\}$ converges strongly to $z \in A^{-1}0$. \square

Let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself. Then $A = I - T$ is an accretive operator which satisfies $C = \overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$ and $A^{-1}0 = F(T)$; see Takahashi [17]. Then, putting $A = I - T$ in Theorem 2, we obtain the following result.

Corollary 3. *Let C be a nonempty closed convex nonexpansive retract of a reflexive Banach space E whose norm is a uniformly Gâteaux differentiable, let P be a nonexpansive retraction of E onto C and let T be a nonexpansive mapping from C into itself. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \frac{1}{1+r_n}x_n + \frac{r_n}{1+r_n}Ty_n, \\ x_{n+1} = P(\alpha_n x + (1-\alpha_n)y_n + f_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} r_n = \infty$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly in $F(T)$.

In the case where A is an m -accretive operator, we obtain the following result.

Corollary 4. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E \times E$ be an m -accretive operator. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)J_{r_n}x_n + f_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n \rightarrow \infty} r_n = \infty$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to an element of $A^{-1}0$.

4. WEAK CONVERGENCE THEOREM

In this section, we prove a weak convergence theorem for Mann's type iteration. Before proving the theorem, we need the following two lemmas.

Lemma 5 (Browder [2]). *Let C be a closed bounded convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $Tz = z$.*

Lemma 6 (Reich [13]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable, let C be a nonempty closed convex subset of E and let $\{T_0, T_1, T_2, \dots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_0$ for all $n \in \mathbb{N}$. Then the set $\bigcap_{n=0}^{\infty} \overline{\text{co}}\{S_m x : m \geq n\} \cap U$ consists of at most one point, where $U = \bigcap_{n=0}^{\infty} F(T_n)$.*

For the proof of Lemma 6, see Takahashi and Kim [20]. Now we can prove the following weak convergence theorem.

Theorem 7. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, let C be a nonempty closed convex nonexpansive retract of E such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$ and let P be a nonexpansive retraction of E onto C . Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = P(\alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n + f_n), \quad n \in \mathbb{N}, \quad (4.1)$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\limsup_{n \rightarrow \infty} \alpha_n < 1$, $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Proof. First we prove the theorem in the case of $f_n \equiv 0$. Let u be an element of $A^{-1}0$ and $y_n = J_{r_n} x_n$. Then for $l = \|x - u\|$, the set $D = C \cap \{z \in E : \|z - u\| \leq l\}$ is a nonempty closed bounded convex subset of E which is invariant under J_s for $s > 0$. Then we may assume that C is bounded. From

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n x_n + (1 - \alpha_n) y_n - u\| \\ &\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \|y_n - u\| \\ &\leq \|x_n - u\|, \end{aligned}$$

$\lim_{n \rightarrow \infty} \|x_n - u\|$ exists. We may assume that $\lim_{n \rightarrow \infty} \|x_n - u\| \neq 0$ without loss of generality. Since A is accretive and E is uniformly convex, it follows that

$$\begin{aligned} \|y_n - u\| &\leq \left\| y_n - u + \frac{r_n}{2} (A_{r_n} x_n - 0) \right\| \\ &= \left\| y_n - u + \frac{1}{2} (x_n - y_n) \right\| \\ &= \left\| \frac{x_n + y_n}{2} - u \right\| \\ &\leq \|x_n - u\| \left\{ 1 - \delta \left(\frac{\|x_n - y_n\|}{\|x - u\|} \right) \right\} \end{aligned}$$

and hence

$$\begin{aligned} (1 - \alpha_n) \|x_n - u\| &\delta \left(\frac{\|x_n - y_n\|}{\|x - u\|} \right) \\ &\leq (1 - \alpha_n) \{ \|x_n - u\| - \|y_n - u\| \} \\ &= \|x_n - u\| - \alpha_n \|x_n - u\| - (1 - \alpha_n) \|y_n - u\| \\ &\leq \|x_n - u\| - \|x_{n+1} - u\| \end{aligned}$$

for all $n \in \mathbb{N}$. Then, by $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\lim_{n \rightarrow \infty} \|x_n - u\| \neq 0$, we obtain $\delta(\|x_n - y_n\|/\|x - u\|) \rightarrow 0$. This implies $x_n - y_n \rightarrow 0$. Let $v \in E$ be a weak

subsequential limit of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. Then it follows that $y_{n_i} \rightharpoonup v$. Further, from

$$\begin{aligned} \|y_n - J_1 y_n\| &= \|(I - J_1)y_n\| = \|A_1 y_n\| \leq \inf\{\|z\| : z \in Ay_n\} \\ &\leq \|A_{r_n} x_n\| = \left\| \frac{x_n - y_n}{r_n} \right\| \end{aligned}$$

and $\liminf_{n \rightarrow \infty} r_n > 0$, we have $y_n - J_1 y_n \rightarrow 0$. Therefore it follows from Lemma 5 that $v \in F(J_1) = A^{-1}0$.

We assume that E has a Fréchet differentiable norm. Putting $T_n = \alpha_n I + (1 - \alpha_n)J_{r_n}$ and $S_n = T_n T_{n-1} \cdots T_0$, we have $\bigcap_{n=0}^{\infty} F(T_n) = A^{-1}0$ and $\{v\} = \bigcap_{n=0}^{\infty} \overline{\text{co}}\{x_m : m \geq n\} \cap A^{-1}0$ by Lemma 6. Therefore $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Next we assume that E satisfies Opial's condition. Let v_1 and v_2 be two weak subsequential limits of the sequence $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. As above, we have $v_1, v_2 \in A^{-1}0$. We claim that $v_1 = v_2$. If not, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - v_1\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - v_1\| < \lim_{i \rightarrow \infty} \|x_{n_i} - v_2\| = \lim_{n \rightarrow \infty} \|x_n - v_2\| \\ &= \lim_{j \rightarrow \infty} \|x_{n_j} - v_2\| < \lim_{j \rightarrow \infty} \|x_{n_j} - v_1\| = \lim_{n \rightarrow \infty} \|x_n - v_1\|. \end{aligned}$$

This is a contradiction. Hence we have $v_1 = v_2$. This implies that $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Finally we prove the theorem in the case of $f_n \not\equiv 0$. Let $U_n z = T_n z + f_n$ for all $z \in E$ and $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ generated by (4.1) satisfies $x_{n+1} = P U_n x_n$. We define, for every $m \in \mathbb{N}$, the sequence $\{z_n(m)\}$ by $z_0(m) = x_m$ and $z_{n+1}(m) = T_{n+m} z_n(m)$, $n \in \mathbb{N}$. Then, from the above discussion, we know that $\{z_n(m)\}$ converges weakly to some $z(m) \in A^{-1}0$ as $n \rightarrow \infty$. By definition, we have

$$\begin{aligned} \|z_n(m+1) - z_{n+1}(m)\| &= \|T_{n+m} T_{n+m-1} \cdots T_{m+1} x_{m+1} - T_{n+m} T_{n+m-1} \cdots T_m x_m\| \\ &\leq \|x_{m+1} - T_m x_m\| \\ &= \|f_m\| \end{aligned}$$

for all $n, m \in \mathbb{N}$. This implies that $\|z(m+1) - z(m)\| \leq \|f_m\|$ for all $m \in \mathbb{N}$. Then, from $\sum_{n=0}^{\infty} \|f_n\| < \infty$, $\{z(m)\}$ is a Cauchy sequence and hence $\{z(m)\}$ converges strongly to some $a \in A^{-1}0$ as $m \rightarrow \infty$. Now we have

$$\begin{aligned} \|x_{n+m+1} - z_{n+1}(m)\| &= \|P U_{n+m} x_{n+m} - P T_{n+m} z_n(m)\| \\ &\leq \|U_{n+m} x_{n+m} - T_{n+m} z_n(m)\| \\ &\leq \|T_{n+m} x_{n+m} - T_{n+m} z_n(m)\| + \|f_{n+m}\| \\ &\leq \|x_{n+m} - z_n(m)\| + \|f_{n+m}\| \\ &\vdots \\ &\leq \sum_{i=m}^{n+m} \|f_i\| \end{aligned}$$

for all $n, m \in \mathbb{N}$. Therefore

$$\begin{aligned} |\langle x_{n+m+1} - a, h \rangle| &\leq |\langle x_{n+m+1} - z_{n+1}(m), h \rangle| + |\langle z_{n+1}(m) - z(m), h \rangle| \\ &\quad + |\langle z(m) - a, h \rangle| \\ &\leq \left(\sum_{i=m}^{n+m} \|f_i\| + \|z(m) - a\| \right) \|h\| + |\langle z_{n+1}(m) - z(m), h \rangle| \end{aligned}$$

for all $h \in E^*$ and $n, m \in \mathbb{N}$. This implies

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\langle x_n - a, h \rangle| &= \limsup_{n \rightarrow \infty} |\langle x_{n+m+1} - a, h \rangle| \\ &\leq \left(\sum_{i=m}^{\infty} \|f_i\| + \|z(m) - a\| \right) \|h\| \end{aligned}$$

for all $h \in E^*$ and $m \in \mathbb{N}$. Letting $m \rightarrow \infty$, we have $\langle x_n - a, h \rangle \rightarrow 0$ for all $h \in E^*$ and hence $\{x_n\}$ converges weakly to $a \in A^{-1}0$. \square

As direct consequences of Theorem 7, we obtain the following two results.

Corollary 8. *Let C be a nonempty closed convex nonexpansive retract of a uniformly convex Banach space E whose norm is Fréchet differentiable or which satisfies Opial's condition, let P be a nonexpansive retraction of E onto C and let T be a nonexpansive mapping of C into itself. Let $x_0 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \frac{1}{1+r_n}x_n + \frac{r_n}{1+r_n}Ty_n, \\ x_{n+1} = P(\alpha_n x_n + (1-\alpha_n)y_n + f_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\limsup_{n \rightarrow \infty} \alpha_n < 1$, $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly in $F(T)$.

Corollary 9. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let $A \subset E \times E$ be an m -accretive operator. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)J_{r_n}x_n + f_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{f_n\} \subset E$ satisfy $\limsup_{n \rightarrow \infty} \alpha_n < 1$, $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=0}^{\infty} \|f_n\| < \infty$. If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

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(S. Kamimura) GRADUATE SCHOOL OF INTERNATIONAL CORPORATE STRATEGY, HITOTSUBASHI UNIVERSITY, 2-1-2 HITOTSUBASHI, CHIYODA-KU, TOKYO 101-8439, JAPAN
E-mail address: kamimura@ics.hit-u.ac.jp

(S. H. Khan, W. Takahashi) DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA, MEGURO-KU, TOKYO 152-8552, JAPAN
E-mail address, S. H. Khan: Safeer.Hussain.Khan@is.titech.ac.jp
E-mail address, W. Takahashi: wataru@is.titech.ac.jp